PROPAGATION OF LONG WAVES OF FINITE AMPLITUDE IN A LIQUID WITH POLYDISPERSED GAS BUBBLES

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Although there exists a large number of analytical and numerical studies of wave processes in liquids with bubbles (for reviews of the basic results and bibliographies, see [1, 2]), relatively little attention has been devoted to wave propagation in polydispersed systems. Propagation of acoustic perturbations in a medium with polydispersed bubbles was considered in [3, 4]. Some numerical and analytical results for the structure of shock waves in polydispersed liquid mixtures with a finite number of bubble fractions were given in [5, 6].

In the present paper we consider propagation of long waves of finite amplitude in liquids with a continuous spectrum of gas bubble sizes using the model of a monodispersed medium. It is shown that in general this description is not correct. However, if the surface tension only slightly affects the pressure inside a bubble and the process is nearly isothermal or adiabatic, then in the long-wavelength limit one can introduce effective bubble radii characterizing the average periods of vibration of the bubbles, heat transfer, and the surface tension on the bubble–liquid interface. Evolution equations are derived for waves of moderate amplitude (weak perturbations of Riemann waves) in the plane, one-dimensional case. These evolutions generalize the Burgers – Korteweg – de Vries (BKdV) equations to larger amplitudes (the BKdV equations for slightly nonlinear waves have been obtained in [1, 2, 7, and 8]).

1. Basic Equations. The unperturbed homogeneous state of an incompressible liquid with bubbles of insoluble gas is denoted with a zero subscript. We assume that in each elementary volume of the medium spherical bubbles are distributed with the density $N_0(a_0)$ in the bubble radius a_0 over the interval $\Delta = [a_-, a_+]$ $(a_0 \in \Delta)$ and $dn_0(a_0) = N_0(a_0)da_0$ is the number of bubbles with radii between a_0 and $a_0 + da_0$ per unit volume of the mixture. Each quasi-monodispersed bubble fraction in the mixture with radii between a_0 and $a_0 + da_0$ is given a Lagrangian index ξ which is treated as a continuous variable. We next assume that the fraction ξ is an independent monodispersed phase in a multiphase continuum [9]. In the perturbed state the parameters of the liquid and of the mixture as a whole do not depend on ξ, but the parameters describing the bubbles will be functions of ξ (they will be given the subscript ξ, where necessary below). For example, $dn_\xi(\mathbf{x}, t) = N_\xi(\mathbf{x}, t)d\xi$, $a_\xi(\mathbf{x}, t)$, $\rho_\xi^0(\mathbf{x}, t)$, $m_\xi(\mathbf{x}, t) = \frac{4}{3}\pi a_\xi^3\rho_\xi^0$, $p_\xi(\mathbf{x}, t)$ are the number of bubbles of fraction ξ per unit volume of the mixture, their radius, density, mass, and gas pressure inside the bubble ($\mathbf{x}$ is the position vector of the elementary volume and t is the time). The parameters of the liquid are given the subscript 1 and the integrated parameters of the bubbles (independent of ξ) are given the subscript 2. Then the equations of motion of a dilute ($\alpha_2 \ll 1$) polydispersed continuum in the one-velocity approximation can be written in the form [1, 3, 9]

$$\begin{gathered} d_t m_\xi = 0,\ d_t\rho_1 + \rho_1\nabla\mathbf{v} = 0\ (d_t\rho + \rho\nabla\mathbf{v} = 0), \\ d_tN_\xi + N_\xi\ \nabla\mathbf{v} = 0,\ \rho d_t\mathbf{v} + \nabla p = \mathbf{0}\ (d_t = \partial_t + \mathbf{v}\nabla,\ \partial_t = \partial/\partial t), \\ \rho_1^0\left[a_\xi d_t^2 a_\xi + \frac{3}{2}(d_t a_\xi)^2 + \frac{4\nu_1}{a_\xi} d_t a_\xi\right] + \frac{2\sigma}{a_\xi} = p_\xi - p, \\ \alpha_2 = \int\limits_\Delta \frac{4}{3}\pi a_\xi^3 N_\xi\, d\xi,\ \rho_2 = \int\limits_\Delta m_\xi N_\xi\, d\xi,\ \alpha_1 = 1 - \alpha_2,\ \rho_1 = \rho_1^0\alpha_1,\ \rho = \rho_1 + \rho_2, \\ \rho_2^0 = \rho_2/\alpha_2. \end{gathered} \tag{1.1}$$

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Here ρ and ρ^0 are the average density and intrinsic density; v is the velocity of the mixture; α is the volume content; ν and σ are the kinematic viscosity and the surface tension; the quantities without subscripts (ρ, $\mathbf{v}$, p) refer to the mixture as a whole.

In the one-velocity approximation we have

$$\rho_1/\rho_{10} = \rho/\rho_0 = N_\xi/N_{\xi 0} \quad (N_\xi = (\rho/\rho_0)N_{\xi 0}), \tag{1.2}$$

and hence the size distribution of the bubbles can be expressed in terms of the initial size distribution.

The system (1.1) is not closed, since it does not contain a relation between p_ξ and a_ξ. This relation can be found from the solution of the heat transfer problem between a single bubble of ideal gas and the surrounding liquid. The solution in the homobaric formulation [$p_g = p_g(t)$], in a coordinate system fixed to the center of the spherical bubble, has the form [1] ($p_g = p_\xi$, $a = a_\xi$)

$$\begin{gathered} p_g(t) = R_2\rho_g(r,t)T_g(r,t),\ w_g(r,t) = \frac{\gamma_2 - 1}{\gamma_2 p_g}\lambda_2\partial_r T_g - \frac{r}{3\gamma_2 p_g}\partial_t p_g, \\ \rho_g c_2(\partial_t T_g + w_g\partial_r T_g) = \lambda_2 r^{-2}\partial_r(r^2\partial_r T_g) + \partial_t p_g, \\ a\partial_t p_g + 3\gamma_2 p_g\partial_t a + 3(\gamma_2 - 1)q_g = 0,\ q_g = -\lambda_2(\partial_r T_g)_{r=a}, \\ T_g|_{r=a} = T_0 = \text{const},\ \partial_r T_g|_{r=0} = 0 \quad (\partial_t = \partial/\partial t,\ \partial_r = \partial/\partial r), \end{gathered} \tag{1.3}$$

where the g subscript denotes gas; r is the radial coordinates; T, w, and q are the temperature, radial velocity, and heat flux on the boundary of the bubble; the quantities R_2, γ_2, c_2, λ_2 are the gas constant, adiabatic index, heat capacity at cosntant pressure, and thermal conductivity of the gas, and are assumed to be constants.

We consider wave propagation with the characteristic period $t_* = L_*/C_*$, where L_* and C_* are the characteristic wavelength and propagation velocity. We introduce the dimensionless variables

$$\begin{gathered} p'_g = \frac{p_g}{p_{g0}},\ p' = \frac{p}{p_0},\ \rho'_g = \frac{\rho_g}{\rho_{g0}},\ \rho' = \frac{\rho}{\rho_0},\ T'_g = \frac{T_g}{T_0},\ \mathbf{v}' = \frac{\mathbf{v}}{C_*}, \\ \mathbf{x}' = \frac{\mathbf{x}}{L_*},\ t' = \frac{t}{t_*},\ a' = \frac{a}{a_0},\ \eta = \frac{r}{a(t)},\ w'_g = \frac{w_g t_*}{a_0},\ q'_g = \frac{q_g a(t)}{\lambda_2 T_0}, \\ \sigma' = \frac{\sigma}{a_0 p_0}\left(p_{g0} = p_0 + \frac{2\sigma}{a_0} = R_2\rho_{g0}T_0,\ a_0 \equiv \xi,\ a'_\xi = a',\ p'_\xi = p'_g\right). \end{gathered} \tag{1.4}$$

Equations (1.1) and (1.3) show that the behavior of the bubble is determined by three different time constants

$$\begin{gathered} \tau_i = a_0\left(\frac{\rho_1^0}{p_0}\right)^{1/2},\ \tau_\lambda = \frac{a_0^2}{\varkappa_2},\ \tau_\mu = \frac{\mu_1}{p_0} \quad \left(\varkappa_2 = \frac{\lambda_2}{\rho_{g0}c_2},\ \mu_1 = \rho_1^0\nu_1\right), \\ \delta_j = \tau_j/t_*,\ j = \lambda, i, \mu \quad (\delta_i = \delta_i(\xi),\ \delta_\lambda = \delta_\lambda(\xi),\ a_0 \equiv \xi), \end{gathered} \tag{1.5}$$

characterizing the inertial, thermal, and viscous effects in the interaction of the bubble with the liquid. The dimensionless parameters δ_j are the ratios of these time constants to the characteristic external time t_*. In the long-wavelength approximation $\delta_i \ll 1$, $\delta_\mu \ll 1$ (see also [10]) and we consider the two cases $\delta_\lambda \ll 1$ and $\delta_\lambda \gg 1$ (the gas inside the bubble is nearly isothermal or adiabatic, respectively).

We introduce the linear averaging operator

$$\langle g\rangle = \left(\int_\Delta gN_{\xi 0}\xi^3\,d\xi\right)\Big/\left(\int_\Delta N_{\xi 0}\xi^3\,d\xi\right) \tag{1.6}$$

(if g does not depend on ξ then $\langle g\rangle = g$) and the average radii [9]

$$a_{m,n} = \left[\left(\int_\Delta N_{\xi 0}\xi^m\,d\xi\right)\Big/\left(\int_\Delta N_{\xi 0}\xi^n\,d\xi\right)\right]^{1/(m-n)},\quad m \neq n. \tag{1.7}$$

Below we will mainly use the dimensionless quantities (1.4) and (1.5) and to simplify the notation we omit the primes on these quantities.

2. Zeroth (Equilibrium) Approximation. Putting $\delta_i = 0$, $\delta_\mu = 0$ and $\delta_\lambda = 0$ ($T_g = 1$, $p_g = \rho_g$) or $\delta_\lambda = \infty$ ($p_g = \rho_g^{\gamma_2}$), we have

$$2[a(\xi)]^{-1}\sigma(\xi) = [a(\xi)]^{-3\varkappa}(1 + 2\sigma(\xi)) - p \quad (a = f_\sigma(\xi, p)). \tag{2.1}$$

Here and below $\varkappa$ is the polytrope index and is equal to 1 in the isothermal case and γ_2 in the adiabatic case; f_σ is the unique positive root of (2.1). Using (1.1), (1.2), and (1.6), we obtain the equation of state for the polydispersed medium

$$\rho = [\alpha_{10} + \alpha_{20}\langle f_\sigma^3(\xi, p)\rangle]^{-1}, \tag{2.2}$$

which is in general different from the equation of state for monodispersed bubbles in a liquid [1], since bubbles of different sizes deform differently. If the bubble radius or the pressure of the liquid is so large that surface tension can be neglected, then the size distribution of the bubbles does not affect the equation of state $\left(\lim\limits_{\sigma\to 0} f_\sigma^3 = f_0^3 = p^{-1/\varkappa}\right)$ and

$$p = [\alpha_{20}\rho/(1 - \alpha_{10}\rho)]^{\varkappa}. \tag{2.3}$$

We consider the first correction to this equation due to surface tension. Expanding (2.1) in a series in σ and keeping only the linear term, find

$$p = \left(\frac{\alpha_{20}\rho}{1-\alpha_{10}\rho}\right)^{\varkappa}\left\{1 + 2\sigma_{3,2}\left[1 - \left(\frac{\alpha_{20}\rho}{1-\alpha_{10}\rho}\right)^{-\varkappa+1/3}\right]\right\}, \tag{2.4}$$

where $\sigma_{3,2} = \sigma/(p_0 a_{3,2})$ [σ is the surface tension and $a_{3,2}$ is given by (1.7)]. Hence we have shown that in the equilibrium approximation to first order in the surface tension the propagation of long waves in a polydispersed mixture will be the same as in a monodispersed mixture with the effective bubble radius $a_{3,2}$ [the case of a monodispersed medium is obtained by putting $N_{\xi 0} = n_0\delta(\xi - a_*)$, $\delta(\xi)$ is the Dirac delta function and a_* is the unperturbed bubble radius. Then $a_{m,n} = a_*$ for any m and n in (1.7)]. Below the surface tension will be neglected for simplicity.

It follows from (2.4) that for small α_{20} perturbation of the pressure p by a small quantity ~ε leads to perturbation in the density ρ by a quantity $\sim\varepsilon\alpha_{20}$. The conservation equations of mass and momentum (1.1) show that the velocity v is perturbed by a quantity $\sim\varepsilon\alpha_{20}$. Converting to dimensionless quantities and neglecting terms $O(\alpha_{20})$, we obtain from (1.1), (1.4), and (2.4)

$$\partial_t r + \nabla\mathbf{u} = 0,\ \partial_t\mathbf{u} + \nabla p = \mathbf{0},\ p = \pi(r) \equiv (1 - r)^{-\varkappa}$$
$$\left(r = \alpha_{20}^{-1}(\rho - 1),\ \mathbf{u} = \alpha_{20}^{-1}\mathbf{v},\ C_* = [p_0/(\alpha_{20}\rho_0)]^{1/2}\right). \tag{2.5}$$

The system of equations (2.5) describes the motion of a barotropic gas and in the plane one-dimensional case has a simple Riemann wave solution. For waves traveling to the right

$$\partial_t r + c(r)\partial_x r = 0,\ c(r) = [\pi'(r)]^{1/2} = [\varkappa(1-r)^{-\varkappa-1}]^{1/2} = U'(r),$$
$$u = U(r),\ U|_{\varkappa=1} = -\ln(1-r),\ U|_{\varkappa\neq 1} = 2\varkappa^{1/2}(\varkappa-1)^{-1}(1-r)^{-(\varkappa-1)/2}. \tag{2.6}$$

Here a prime denotes differentiation with respect to the argument; c(r) is the constant density propagation velocity, which can be shown to obey the equation

$$\partial_t c + c\partial_x c = 0. \tag{2.7}$$

When $\varkappa = 1$ the pressure of the mixture $p = \pi(r) = c(r)$ satisfies (2.7).

3. Interphase Heat Transfer. In terms of the variables (1.4) $[\partial_r = a^{-1}\partial_\eta,\ \partial_t = \partial_t - a^{-1}(\partial_t a)\eta\partial_\eta]$ the heat problem (1.3) reduces to

$$a^2(p_g\partial_t T_g - \gamma_2^{-1}(\gamma_2 - 1)T_g\partial_t p_g) = \delta_\lambda^{-1}[T_g\eta^{-2}\partial_\eta(\eta^2\partial_\eta T_g) - (\partial_\eta T_g + \eta q_g)\partial_\eta T_g],$$
$$3\gamma_2 p_g a\,\partial_t a + a^2\,\partial_t p_g + 3\gamma_2\delta_\lambda^{-1} q_g = 0,\ q_g = -\partial_\eta T_g|_{\eta=1},\ T_g|_{\eta=1} = 1. \tag{3.1}$$

We consider thick ($\delta_\lambda \ll 1$) and thin ($\delta_\lambda \gg 1$) thermal boundary layers in the bubble.

A. $\delta_\lambda \ll 1$. We expand T_g and q_g in asymptotic series in the small parameter δ_λ:

$$T_g = T_{g0} + \delta_\lambda T_{g1} + \ldots,\ q_g = q_{g0} + \delta_\lambda q_{g1} + \ldots,\ q_{gj} = -\partial_\eta T_{gj}|_{\eta=1},\ j = 0, 1 \ldots \tag{3.2}$$

Substituting (3.2) into (3.1), we obtain $T_{g0} \equiv 1$, $q_{g0} \equiv 0$ and

$$\eta^{-2}\partial_\eta(\eta^2\partial_\eta T_{g1}) = -(1-\gamma_2^{-1})a^2\partial_t p_g,\ \eta^{-2}\partial_\eta(\eta^2\partial_\eta T_{g2}) =$$
$$= -T_{g1}\eta^{-2}\partial_\eta(\eta^2\partial_\eta T_{g1}) + (\partial_\eta T_{g1} + \eta q_{g1})\partial_\eta T_{g1} + a^2(p_g\partial_t T_{g1} -$$
$$-(1-\gamma_2^{-1})T_{g1}\partial_t p_g),\ T_{gj}|_{\eta=1} = 0,\ j = 1, 2, \ldots \tag{3.3}$$

Solving (3.3) simultaneously, we find

$$T_{g1}=\frac{1}{6}\left(1-\gamma_2^{-1}\right)(1-\eta^2)a^2\,\partial_t p_g,\quad q_{g1}=\frac{1}{3}\left(1-\gamma_2^{-1}\right)a^2\,\partial_t p_g,$$

$$q_{g2}=-\frac{1}{15}a^2 p_g\,\partial_t q_{g1}.$$

Substituting the expressions for q_{g1} and q_{g2} into (3.2) and into the energy integral (3.1), and integrating with respect to t in the linear approximation in δ_λ, we have

$$p_g=a^{-3}\left[1-\frac{1}{5}\left(1-\gamma_2^{-1}\right)\delta_\lambda a^{-2}\,\partial_t a\right]=a^{-3}\left[1-\delta_\lambda F(a,t)\right]. \tag{3.4}$$

This is the required relation between p_g and a, in the case when the behavior of the bubble is nearly isothermal. We note that (3.4) can be obtained using a different method by introducing the Nusselt number for the interior problem $Nu_g = 10$ [1]. However in [1] the value $Nu_g = 10$ was found from the solution of the linear problem of small harmonic vibrations of the bubble with a frequency approaching zero.

B. $\delta_\lambda \gg 1$. In most of the bubbles the temperature follows the adiabatic law $T_g = p_g^{1-1/\gamma_2}$ and its spatial variation is confined to a narrow layer of thickness $\sim\delta_\lambda^{-1/2}$ near the boundary. Introducing the variable $\zeta = \delta_\lambda^{1/2}(1-\eta)$ and letting $\chi = p_g^{-1+1/\gamma_2}T_g$, in the zeroth approximation we obtain from (3.1)

$$\begin{gathered}a^2 p_g^{1/\gamma_2}\,\partial_t\chi=\chi\,\partial_\zeta^2\chi+(q-\partial_\zeta\chi)\,\partial_\zeta\chi,\quad \chi|_{\zeta=0}=p_g^{-1+1/\gamma_2},\quad \chi|_{\zeta=\infty}=1,\\ q=\partial_\zeta\chi|_{\zeta=0}=\delta_\lambda^{-1/2}p_g^{-1+1/\gamma_2}q_g,\\ p_g^{-1}\partial_t p_g+3\gamma_2 a^{-1}\partial_t a=-3\gamma_2\delta_\lambda^{-1/2}p_g^{-1/\gamma_2}a^{-2}q.\end{gathered} \tag{3.5}$$

It is evident that the quantity q is of order unity and depends on a, p_g, and t. Hence we find from the energy integral $p_g=a^{-3\gamma_2}\left(1+O\left(\delta_\lambda^{-1/2}\right)\right)$ [it then follows that $a^2p_g^{1/\gamma_2}=a^{-1}+O\left(\delta_\lambda^{-1/2}\right)$ and $p_g^{-1+1/\gamma_2}=a^{3(\gamma_2-1)}+O\left(\delta_\lambda^{-1/2}\right)$ and these relations can be used in the coefficient in front of $\partial_t\chi$ and in the boundary condition for $\zeta = 0$ in (3.5)] and

$$p_g=a^{-3\gamma_2}\left(1-\delta_\lambda^{-1/2}F(a,t)\right),\quad F=3\gamma_2\int\limits_0^t a(t)\,q(a(t),t)\,dt. \tag{3.6}$$

It is a very complicated problem to obtain an explicit expression for F. If the temperature inside the bubble does not vary very sharply [the quantity p_g^{-1+1/γ_2} is close to unity, which occurs for relatively small amplitudes or when the adiabatic index γ_2 is close to unity (a "heavy" gas)] we can put $t \to \theta$ and linearize (3.5) in χ and then we obtain a linear heat conduction equation in the coordinates (θ, ζ) and we can calculate F as

$$F=-\frac{3\gamma_2}{\sqrt{\pi}}\int\limits_0^t\frac{\left[a^{3(\gamma_2-1)}(\tau)-1\right]a(\tau)\,d\tau}{\left[\theta(t)-\theta(\tau)\right]^{1/2}},\quad \theta(t)=\int\limits_0^t a(t)\,dt. \tag{3.7}$$

If in addition the amplitude of the perturbation in a is small ($|a-1| \ll 1$) then

$$F=-\frac{9\gamma_2(\gamma_2-1)}{\sqrt{\pi}}\int\limits_0^t\frac{[a(\tau)-1]\,d\tau}{(t-\tau)^{1/2}}. \tag{3.8}$$

In addition to the methods described above, there are other ways of approximately taking into account interphase heat transfer in a medium with bubbles. An example is the use of an effective viscosity [1]. This method uses the solution of the linear heat transfer problem for a bubble and from the damping decrement of the normal vibrations of the bubble one artificially introduces an effective viscosity in the Rayleigh–Lamb equation [1]

$$\nu_{\text{eff}}=\nu_1+\frac{3(\gamma_2-1)}{4\sqrt{2}}a_0^{1/2}\varkappa_2^{1/2}\left(\frac{3\varkappa p_0}{\rho_1^0}\right)^{1/4}. \tag{3.9}$$

Using ν_{eff} as an effective viscosity, τ_μ in (1.5) will depend on ξ ($a_0 = \xi$) [the quantities in (2.9) are dimensional].

4. First Approximation. Using (3.4) and (3.6), the dimensionless Rayleigh–Lamb equation is written in the form

$$p = a_\xi^{-3\varkappa}\left\{1 - a_\xi^{3\varkappa}\left[\delta_i^2\left(a_\xi d_t^2 a_\xi + \frac{3}{2}(d_t a_\xi)^2\right) + 4\delta_\mu a_\xi^{-1} d_t a_\xi + \delta_T a_\xi^{-3\varkappa} F(a_\xi, t)\right]\right\}. \tag{4.1}$$

Here $\delta_T = \delta_\lambda^{-1/2}$ when $\delta_\lambda \gg 1$ and $\delta_T = \delta_\lambda$ when $\delta_\lambda \ll 1$ and $F(a_\xi, t)$ corresponds to F from (3.6) in the first case and to F from (3.4) in the second.

We see from (4.1) that $a_\xi = A\left[1 + O(\delta_i^2, \delta_\mu, \delta_T)\right]$, $A = p^{-1/(3\varkappa)}$. Putting $\delta_i^2 \sim \delta_\mu \sim \delta_T \sim \delta$ and substituting this expression for a_ξ in the terms $\sim\delta$ in (4.1), we find, neglecting terms $\sim\delta^2$

$$a_\xi^3 = p^{-1/\varkappa}\left\{1 - \frac{1}{\varkappa} p^{-1}\left[\delta_i^2\left(A d_t^2 A + \frac{3}{2}(d_t A)^2\right) + 4\delta_\mu A^{-1} d_t A + \delta_T A^{-3\varkappa} F(A, t)\right]\right\}. \tag{4.2}$$

We next apply the operator ⟨ ⟩ (1.6) to both sides of (4.2). According to (1.1) and (1.2) we have $\alpha_2/(\alpha_{20}\rho)$ on the left-hand side, while the right-hand side keeps the same form, in view of the linearity of the averaging operator and the fact that p and A are independent of ξ. The only change is that the average values $\langle\delta_i^2\rangle$, $\langle\delta_\mu\rangle$, and $\langle\delta_T\rangle$ appear in place of the corresponding values of δ. These average values can be expressed in terms of average radii defined by (1.7). Indeed, from (1.5), (3.4), (3.6), and (3.9) we find

$$\begin{aligned}
&\delta_j^* = \tau_j^*/t_*,\ j = i, \mu, T,\ \langle\tau_i^2\rangle = (\tau_i^*)^2,\ \tau_i^* = a_{5,3}(\rho_1^0/p_0)^{1/2},\\
&1)\ \langle\tau_\mu\rangle = \tau_\mu^* = \tau_\mu,\ \langle\tau_T\rangle = \tau_\lambda^*,\ \tau_\lambda^* = a_{5,3}^2/\varkappa_2,\\
&2)\ \langle\tau_\mu\rangle = \tau_\mu^* = \tau_\mu,\ \langle\tau_T\rangle = t_*^{3/2}(\tau_\lambda^*)^{-1/2},\ \tau_\lambda^* = a_{3,2}^2/\varkappa_2,\\
&3)\ \tau_T \equiv 0,\ \langle\tau_\mu\rangle = \tau_\mu^* = \nu_{\text{eff}}^*\rho_1^0/p_0,\ \nu_{\text{eff}}^* = \nu_1 + \frac{3(\gamma_2 - 1)}{4\sqrt{2}} a_{\frac{7}{2},3}^{1/2}\varkappa_2^{1/2}\left(\frac{3\varkappa p_0}{\rho_1^0}\right)^{1/4},
\end{aligned} \tag{4.3}$$

where the cases 1, 2, 3 correspond to heat transfer with thick $\delta_\lambda \ll 1$, $\varkappa = 1$) and thin ($\delta_\lambda \gg 1$, $\varkappa = \gamma_2$) thermal boundary layers and to heat transfer with an effective viscosity.

Therefore to terms $O(\delta^2)$ (4.2) can be rewritten in the form of an equation of state

$$\begin{aligned}
p &= A^{-3\varkappa} - \left[(\delta_i^*)^2\left(A d_t^2 A + \frac{3}{2}(d_t A)^2\right) + 4\delta_\mu^* A^{-1} d_t A + \delta_T^* A^{-3\varkappa} F(A, t)\right],\\
A &= [(1 - \alpha_{10}\rho)/(\alpha_{20}\rho)]^{1/3} \approx (1 - r)^{1/3},
\end{aligned} \tag{4.4}$$

which together with the conservation of mass and momentum equations for the mixture [for example, the first two equations of (2.5) with $d_t \approx \partial_t$] forms a closed system of equations. In other words, if the effect of surface tension is negligibly small, then in the first approximation in the small parameter δ we have 1) when $\delta_\lambda \ll 1$ propagation of long waves in a polydispersed mixture is analogous to propagation of waves in a monodispersed system with the same physical parameters of the two phases, the same volume fractions in the mixture, and with the effective bubble radius $a_{5,3}$; 2) when $\delta_\lambda \gg 1$ and in the effective viscosity method such a direct analogy does not exist since heat transfer is determined by a smaller effective bubble radius than the inertial effects ($a_{3,2} \leqslant a_{7/2,3} \leqslant a_{5,3}$, this can be shown with the help of Hölder's inequality [9]). Nevertheless wave propagation in a polydispersed mixture can be described in terms of the model of a monodispersed medium using corrections for polydispersion in the coefficients. The bubble radius in the model of a monodispersed medium is then a function of the density [A, see (4.4)].

5. Perturbation of Riemann Waves. We consider (2.5) and (4.4) in the case of plane one-dimensional waves

$$\partial_t r + \partial_x u = 0,\ \partial_t u + \partial_x p = 0,\ p = \pi(r) - \delta\varphi(t, r, \partial_t r, \partial_t^2 r),\ \delta \ll 1. \tag{5.1}$$

When $\delta = 0$ a solution exists in the form of the simple wave (2.6), which can be represented as $\psi = \Psi(x - c(r)t]$ (ψ is the any of the required functions). We assume that the correction $\sim\delta$ in the equation of state perturbs the solution by a quantity $\sim\delta$ [$u = U(r) + \delta u_1(x, t)$, where $u_1(x, t) \sim 1$]. Then one expects that the perturbation leads to a "slow" (time scale δt) time dependence of the Riemann solution in a co-moving coordinate system $\psi = \Psi(x - c(r)t, \delta t)$, i.e., u_1 satisfies the equation $\partial_t u_1 + c(r)\partial_x u_1 = O(\delta)$. Substituting the above expression for u into the first equation of (5.1) and adding the result to the second equation dividing by c(r), we have from (2.6) and (5.1)

$$\partial_t r + c(r)\partial_x r - \frac{1}{2}\delta[c(r)]^{-1}\partial_x\varphi(t, r, \partial_t r, \partial_t^2 r) = 0. \tag{5.2}$$

The time derivatives appearing in the equation for φ can be replaced by derivatives with respect to the spatial variable x:

$$\partial_t r = -c(r)\,\partial_x r + O(\delta),\ \partial_t^2 r = c^2(r)\,\partial_x^2 r + 2c\,(r)\,c'(r)\,(\partial_x r)^2 + O(\delta).$$

The residual term here is not significant, since (5.2) was obtained in the linear approximation in δ. In addition, in (5.2) we can substitute a Riemann wave $u = U(r)$, $p = \pi(r)$ and find wave equations for u and p. For example, using (4.4), we have for p

$$\partial_t p + \varkappa^{1/2} p^{(\varkappa+1)/(2\varkappa)} \partial_x \Big\{ p + \frac{1}{6} (\delta_i^*)^2\, p^{-1+1/(3\varkappa)} \Big[p\partial_x^2 p + \frac{1}{6\varkappa} (\partial_x p)^2 \Big] - \\ - \frac{2}{3}\,\varkappa^{-1/2} \delta_\mu^* p^{-(\varkappa-1)/(2\varkappa)} \partial_x p - \frac{1}{2}\,\delta_T^* p F\left(p^{-1/(3\varkappa)}, t\right) \Big\} = 0. \tag{5.3}$$

In general (5.3) is an integrodifferential equation (because of F) but it becomes a differential equation if we assume the effective viscosity approximation ($\delta_T^* = 0$) or if we assume that the behavior of the bubbles is nearly isothermal ($\varkappa = 1,\ \delta_T^* = \delta_\lambda^*,\ \delta_\mu^* = \delta_\mu$). In the latter case

$$\partial_t p + p\partial_x \Big\{ p + \frac{1}{6} (\delta_i^*)^2\, p^{-2/3} \Big[p\partial_x^2 p + \frac{1}{6} (\partial_x p)^2 \Big] - \frac{2}{3}\,\delta_\mu \partial_x p - \frac{1}{30}\,\delta_\lambda^* \left(1 - \gamma_2^{-1}\right) p^{4/3} \partial_x p \Big\} = 0. \tag{5.4}$$

This equation takes on a particularly simple form when $(\delta_i^*)^2 \ll \delta_\mu$, $\delta_\lambda^* \ll \delta_\mu$ (this occurs in the case of small bubbles in a very viscous liquid):

$$\partial_t p + p\partial_x p - \frac{2}{3}\,\delta_\mu p \partial_x^2 p = 0. \tag{5.5}$$

6. εδ Approximation. We assume that the relative amplitude of the pressure perturbation is small ($|p - 1| \sim \varepsilon \ll 1$) and $\delta \sim \varepsilon^n$, $n > 0$. Neglecting terms $\sim\varepsilon\delta$ ($\sim\varepsilon^{n+1}$) in comparison with unity [linearizing the dissipative-dispersion terms in equations like (5.2)], we can write (5.3) in the form

$$\partial_t p + \varkappa^{1/2} p^{\beta} \partial_x p - \frac{1}{2}\,\varkappa^{1/2} \delta_T^* \partial_x F\Big(1 - \frac{1}{3\varkappa}(p-1), t\Big) - \frac{2}{3}\,\delta_\mu^* \partial_x^2 p + \frac{1}{6}\,\varkappa^{1/2} (\delta_i^*)^2\, \partial_x^3 p = 0 \\ (\beta(\varkappa) = (\varkappa + 1)/(2\varkappa)). \tag{6.1}$$

When $\varkappa = 1$, $\delta_\lambda^* \ll 1$ we obtain from (5.4) and (6.1) the BKdV equations

$$\partial_t p + p\partial_x p - \frac{2}{3}\,\delta_{\mu\lambda}^* \partial_x^2 p + \frac{1}{6}\,(\delta_i^*)^2\, \partial_x^3 p = 0, \\ \delta_{\mu\lambda}^* = \delta_\mu + \frac{1}{20}\left(1 - \gamma_2^{-1}\right) \delta_\lambda^*\ \ \left(\delta_T^* = 0,\ \nu_{\mathrm{eff}}^* = \nu_1 + \frac{1}{20}\left(1 - \gamma_2^{-1}\right) p_0 a_{5,3}^2/(\rho_1^0 \varkappa_2)\right), \tag{6.2}$$

which can also be obtained using the effective viscosity method ($\delta_T^* = 0$, $\nu = \nu_{\mathrm{eff}}^*$), where ν_{eff}^* differs from the effective viscosity introduced in (4.3).

When $\varkappa = \gamma_2, \delta_\lambda^* \gg 1$ one can use the linear solution (3.8) for F in (6.1). Replacing the differentiation with respect to x by differentiation with respect to t and assuming that the process evolves in time from $-\infty$, (6.1) takes the form

$$\partial_t p + \gamma_2^{1/2} p^{\beta} \partial_x p + \frac{3(\gamma_2 - 1)}{2\sqrt{\pi\delta_\lambda^*}} \int_{-\infty}^{t} \frac{\partial p}{\partial \tau} \frac{d\tau}{\sqrt{t-\tau}} - \frac{2}{3}\,\delta_\mu \partial_x^2 p + \frac{1}{6}\,\gamma_2^{1/2} (\delta_i^*)^2\, \partial_x^3 p = 0,\ \beta = \beta(\gamma_2). \tag{6.3}$$

Assuming $\delta \sim \varepsilon$ ($n = 1$) the nonlinearity can be taken into account in the quadratic approximation, as is done in the usual method of deriving the slightly nonlinear long-wavelength equations [11]. Then (6.3) is similar in form to the integrodifferential equation of [2]; however, in [2] the kernel of the integral was an exponential, which corresponds to the case $\delta_\lambda^* \ll 1$. In the effective viscosity method one puts $\gamma_2 = \varkappa$, $\delta_\lambda^* = \infty$, and $\delta_\mu = \delta_\mu^*$ [see (4.3)] into (6.3); then in the quadratic approximation (6.3) reduces to the BKdV equations [1, 2].

7. Discussion. The procedure for obtaining asymptotic solutions in the limit $\delta \to 0$ in Secs. 4-6 assumes that the derivatives are finite [$\partial_t^n a = O(1)$, $\partial_x^n p = O(1)$], which is true for all x and t when the wave spectrum does not contain high-frequency harmonics (for example, in systems with strong dissipation for rarefaction waves, weak long waves, long waves in the initial stages of evolution, and so on). However one would not expect this assumption to be valid for compression waves with amplitudes $\varepsilon \gtrsim 1$, since the scale of the wave structure is also determined by dispersion and dissipative effects, which are supposed to be small according to the expansions used here (actually the wave structrures correspond to regions where the asymptotic expansions are nonuniform). Nevertheless the effective bubble sizes found

above can be used to estimate the dimensions of these regions for compression waves of moderate amplitude in a polydispersed mixture using the known results for a monodispersed mixture [1].

The equations of the $\varepsilon\delta$ approximation are of interest, since they are intermediate between the equations of the quadratic approximation in ε and the equations for simple waves. The $\varepsilon\delta$ approximation fully takes into account the hydrodynamic nonlinearity and the nonlinearity of the equilibrium equation of state, therefore it can be used even for large-amplitude waves, when the equilibrium approximation ($\delta = 0$) is justified. On the other hand, the equations of the $\varepsilon\delta$ approximation are no less rigorous than the equations of the ε^2 approximation and can be used to describe slightly nonlinear waves with dispersion and dissipation. We note that the equation for c [see (2.7)] is the same in the ε^2 and $\varepsilon\delta$ approximations [for example, (6.2)].

LITERATURE CITED

1. R. I. Nigmatulin, Dynamics of Multiphase Media [in Russian], Part II, Nauka, Moscow (1987).
2. V. E. Nakoryakov, B. G. Pokusaev, and I. R. Shreiber, Wave Propagation in a Gas–Liquid Medium [in Russian], Inst. of Heat Physics, Siberian Branch, Russian Academy of Sciences, Novosibirsk (1983).
3. V. K. Kedrinskii, "Propagation of disturbances in a liquid containing gas bubbles," Prikl. Mekh. Tekh. Fiz., No. 4 (1968).
4. V. Sh. Shagapov, "Propagation of small disturbances in liquids with bubbles," Prikl. Mekh. Tekh. Fiz., No. 1 (1977).
5. V. Sh. Shagapov, "Structure of shock waves in a polydispersed mixture of gas bubbles in a liquid," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1976).
6. S. L. Gavrilyuk, "Existence, uniqueness, and stability of travelling waves in a medium with polydispersed bubbles in the presence of dissipation," in: Partial Differential Equations [in Russian], Inst. of Mathematics, Siberian Branch, Russian Academy of Sciences, Novosibirsk (1989).
7. L. van Wijngaarden, "One-dimensional flow of liquids containing small gas bubbles," Ann. Rev. Fluid Mech., 4 (1972). Rhleology of Suspensions [Russian translation], Mir, Moscow (1975).
8. V. E. Nakoryakov, V. V. Sobolev, and I. R. Shreiber, "Long-wavelength disturbances in gas–liquid mixtures," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5 (1972).
9. N. A. Gumerov and A. I. Ivandaev, "Sound propagation in polydispersed suspensions of matter in gas," Prikl. Mekh. Tekh. Fiz., No. 5 (1988).
10. N. A. Gumerov, "Long waves of finite amplitude in polydispersed suspensions of matter in gas," Prikl. Mekh. Tekh. Fiz., No. 4 (1990).
11. O. V. Rudenko and S. I. Soluyan, Theoretical Foundations of Nonlinear Acoustics [in Russian], Nauka, Moscow (1975).